Minimum Realizations and System Modeling. I. Fundamental Theory and Algorithms

This review article surveys representative literature on minimum realizations and system modeling. This theory makes possible the construction of minimum size state-space models directly from experimental input/output data. Theoretical developments, numerical algorithms, and the connection with other methods of identification of linear systems are covered. Partial minimum realizations which are extensions of the concepts developed are also included. A numerical example illustrates many of the techniques.

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SCOPE

One of the most important and interesting problems of concern to the chemical engineer is the modeling or identification of real dynamical systems. Such modeling must be done in the face of significant nonlinearities and high dimensionality, both items being major components in chemical engineering systems. The results of such analysis can, of course, be used in a descriptive and/or predictive sense for optimizing and controlling the actual behavior of a system.

There currently exists a relevant theory for linear, constant, dynamical systems which makes possible the construction of state-space models directly from input/output experimental data. This theory, termed realization theory, has generated a more fundamental understanding of the relationships between state-space, transfer function, and input/output descriptions of linear systems.

Underlying many, if not all, of the important principles of realization theory are the dual concepts of controllability and observability. The importance of these concepts, first recognized by Kalman (1963) and Gilbert (1963), in the construction of state-space representations of minimum dimension and in the stability analysis of physical processes has been documented (Kalman, 1965a, b; Roberts, 1969). Techniques developed within the last few years have allowed the extension to the construction of linear, constant-coefficient, dynamical models of nonlinear systems. Although subject to many of the same limitations that restrict other linearization procedures, these techniques have proven to be quite useful in the modeling of many varieties of chemical engineering processes (Kallina, 1970; Rossen, 1972).

In this paper, realization techniques which operate in the state or output space and yield linear models from input/output data will be developed. The methods will not only furnish a linear representation of a system but will also develop a minimal model in the sense of specifying the minimum number of system parameters, a minimal realization. The black box approach is used in these methods because only the input/output data are used in the construction of the models.

It is important to realize that such model construction methods can form the basis for a rational choice of model size for real linear systems. In the second paper in this series, new developments which can be applied to nonlinear lumped and distributed parameter systems will be outlined. As such, the theory forms a viable algorithm for real chemical engineering systems.

CONCLUSIONS AND SIGNIFICANCE

To help the reader understand what is a most complex analysis, some of the basic principles of linear theory are reviewed as they apply to linear realizations. This review includes the concepts of controllability and observability which are fundamental features of all of the developments. Then some of the computer algorithms and the associated documentation are detailed. Special emphasis is placed on the algorithm developed by Ho and Kalman (1966) which allows the construction of minimum realizations directly from the experimental input/output response curves. This algorithm is analyzed in detail because it forms the basis for much subsequent work in the area.

The algorithm developed by Tether (1970) for the partial realization of incomplete input/output data is also detailed. The area of linear system modeling via the theory of minimum partial realizations is then related to the frequently encountered moment methods and frequency response techniques A numerical example is presented to show how a state-space minimum model may be constructed either by proceeding through a transfer function or through input/output response data directly.

LINEAR SYSTEM THEORY

Because of the fundamental link between linear systems and the realization theory, it is appropriate to begin this discussion with a brief review of linear system theory. In the continuous case, the linear, constant-coefficient, finite-dimensional, dynamic system can be represented as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{H}\mathbf{x}(t) \end{aligned} \tag{1}$$

Likewise, in the discrete case, we find that

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{H}\mathbf{x}(k) \end{aligned} \tag{2}$$

In these equations $\mathbf{x}(t)$, $\mathbf{u}(t)$, and $\mathbf{y}(t)$ designate the $(n \times 1)$ state vector, the $(p \times 1)$ control or input vector, and the $(m \times 1)$ output or measurement vector, respectively.

For the remainder of the paper, only the continuous-time case will be discussed. Remember, however, that the discrete case is analogous and the treatments will be applicable to the theory of discrete-time minimum realizations $\{\mathbf{\Phi}, \mathbf{G}, \mathbf{H}\}$, as well as continuous-time solutions $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$.

The triple of constant matrices $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$ which represents the internal operation of the linear system can be used to determine a priori the system's response to any input. Such a representation is called the *realization* of the system. Any system has an infinite number of realizations which will predict the identical response for any particular input.

To illustrate this last point let us define a new vector $\boldsymbol{\eta}$ such that

$$\boldsymbol{\eta} = \mathbf{T}\mathbf{x} \quad \text{or} \quad \mathbf{x} = \mathbf{T}^{-1}\boldsymbol{\eta}$$

where $\mathbf{T}$ is any nonsingular square matrix. Substitution into (1) yields

$$\begin{aligned} \mathbf{T}^{-1}\dot{\boldsymbol{\eta}} &= \mathbf{F}\mathbf{T}^{-1}\boldsymbol{\eta} + \mathbf{G}\mathbf{u} \\ \mathbf{y} &= \mathbf{H}\mathbf{T}^{-1}\boldsymbol{\eta} \end{aligned}$$

or, upon rearrangement,

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{T}\mathbf{F}\mathbf{T}^{-1}\boldsymbol{\eta} + \mathbf{T}\mathbf{G}\mathbf{u} \\ \mathbf{y} &= \mathbf{H}\mathbf{T}^{-1}\boldsymbol{\eta} \end{aligned} \tag{3}$$

The effect of inputs $\mathbf{u}(t)$ on $\mathbf{y}(t)$ will be the same for this new system (3) as for the original system (1). Thus the triple $\{\mathbf{T}\mathbf{F}\mathbf{T}^{-1}, \mathbf{T}\mathbf{G}, \mathbf{H}\mathbf{T}^{-1}\}$ will also be a realization of (1) and the predicted responses using this realization will be identical to those predicted by $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$. Since there are an infinite number of nonsingular matrices $\mathbf{T}$, there are an infinite number of such realizations.

Note also that the transfer function matrix for (1) is

$$\overline{\mathbf{Y}}(s) = \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$$

while that for (3) is

$$\overline{\mathbf{Y}}_1(s) = \mathbf{H}\mathbf{T}^{-1}(s\mathbf{I} - \mathbf{T}\mathbf{F}\mathbf{T}^{-1})^{-1}\mathbf{T}\mathbf{G}$$

By algebraic manipulation it can be shown that $\overline{\mathbf{Y}}_1(s) = \overline{\mathbf{Y}}(s)$. Thus the two systems are related by the linear transformation given by $\mathbf{T}$, are algebraically equivalent, and have the same transfer function matrix.

The dimension of the realization will be defined as the dimension n of the $\mathbf{F}$ matrix. Different realizations of the same system can be of different dimension but for convenience in computation as well as for simplicity of the solution, we would like to obtain a realization of the smallest dimension, a minimum dimension or *minimum realization*.

The solution to (1) can be represented as (with $t(0) = 0$)

$$\mathbf{x}(t) = \exp[\mathbf{F}t]\,\mathbf{x}(0) + \int_0^t \exp[\mathbf{F}(t-\tau)]\,\mathbf{G}\mathbf{u}(\tau)\,d\tau \tag{4}$$

If the initial condition is

$$\mathbf{x}(0) = \mathbf{0} \tag{5}$$

and, since $\mathbf{y}(t) = \mathbf{H}\mathbf{x}(t)$, we find that

$$\mathbf{y}(t) = \mathbf{H}\mathbf{x}(t) = \mathbf{H}\int_0^t \exp[\mathbf{F}(t-\tau)]\,\mathbf{G}\mathbf{u}(\tau)\,d\tau \tag{6}$$

This is the system's response to any input $\mathbf{u}(\tau)$. To obtain the response to an impulse $\delta(t)$ in one of the input variables, we can substitute $\delta(t = 0)$ for the appropriate element in the vector $\mathbf{u}(\tau)$. When this is done for each input element, the results are combined to obtain the impulse or pulse response function matrix

$$\mathbf{Y}(t) = \mathbf{H}\exp[\mathbf{F}t]\,\mathbf{G} \tag{7}$$

In the Laplace transform domain

$$\overline{\mathbf{Y}}(s) = \mathbf{H}\int_0^\infty \int_0^t \exp[\mathbf{F}(t-\tau)]\,\mathbf{G}\mathbf{u}(\tau)\,d\tau\,\exp(-st)\,dt \tag{8}$$

and the equivalent result is

$$\overline{\mathbf{Y}}(s) = \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} \tag{9}$$

where $\overline{\mathbf{Y}}(s)$ is the transform of $\mathbf{Y}(t)$.

As a basis for mathematical model building, the impulse is attractive because it is simple enough to be implemented or approximated in most physical systems and because, for stable systems, the pulse response damps out, returning the system to the initial state. Other inputs tend to alter the steady state of the system, thereby disturbing the system they were meant to model.

Markov Parameters

Each of these relations can be characterized by a sequence of constant matrices $\mathbf{Y}_k$, where $\mathbf{Y}_k = \mathbf{H}\mathbf{F}^k\mathbf{G}$, $k = 0, 1, 2, \ldots$. These matrices, known as *Markov parameters*, can be obtained from the experimental input/output data by a power series time fit of the expansion of (7) in terms of t^k,

$$\mathbf{Y}(t) = \sum_{k=0}^{\infty} \mathbf{Y}_k t^k/k! \tag{10}$$

If the Laplace transform is given, the parameters can be obtained by an expansion of (9) in powers of $1/s$

$$\overline{\mathbf{Y}}(s) = \sum_{k=1}^{\infty} \mathbf{Y}_{k-1}\, s^{-k} \tag{11}$$

Using the $\mathbf{Y}_k$ we can transform each of these separate cases into a problem of the same form. If two solutions are characterized by the same set of $\mathbf{Y}_k$, $k = 0, 1, 2, \ldots$ they are equivalent. Since other input responses can be completely determined by the $\mathbf{Y}_k$, two equivalent systems will predict exactly the same response to inputs other than the impulse function. The $\mathbf{Y}_k$ can be used to calculate a priori the system's response to these inputs. Therefore, any triple which produces the same sequence of $\mathbf{Y}_k$ as the system in question will be a realization of that system.

CONTROLLABILITY AND OBSERVABILITY

The construction of realizations of minimum dimension relies heavily on the concepts of controllability and ob-

servability. The important pioneering works of Kalman (1963) and Gilbert (1963) should be consulted for a detailed analysis; here we merely present a brief summary of certain basic results. In this discussion, the theorem approach will be used so that the main features can be highlighted.

Controllability

A system is said to be completely controllable or completely state-controllable if any state of the system can be reached from any initial state of the system in a finite interval of time by the use of some control action. Equation (4) can be rearranged to show that the system is completely state-controllable iff (if and only if)

$$\mathbf{x}(t_0) = \exp[\mathbf{F}(t_0 - t_f)]\,\mathbf{x}(t_f) - \int_{t_0}^{t_f} \exp[\mathbf{F}(t_0 - \tau)]\,\mathbf{G}\mathbf{u}(\tau)\,d\tau \quad (12)$$

for some $\mathbf{u}(\tau)$. The necessary and sufficient conditions for complete state-controllability for (1) can be stated in the following theorem [Ho and Kalman, (1963)]:

Theorem 1. "A linear, constant, dynamical, finite dimensional, continuous-time system is completely state-controllable if the block matrix

$$[\mathbf{G}\ \mathbf{FG}\ \mathbf{F}^2\mathbf{G}\ \ldots\ \mathbf{F}^{n-1}\mathbf{G}] \text{ has rank } n."$$

The proof of this theorem is dependent on the Cayley-Hamilton theorem which states that a square matrix $\mathbf{F}$ satisfies its characteristic equation.

Since $\mathbf{x}(t)$ cannot in general be measured directly, it will be useful to introduce the concept of output-controllability. A linear system is said to be completely output-controllable if any output $\mathbf{y}(t_0)$ can be transferred to any other output $\mathbf{y}(t_f)$ in a finite interval of time $[t_0, t_f]$ by some control $\mathbf{u}(\tau)$.

Theorem 2. "Any linear, finite-dimensional, continuous-time, constant dynamical system is completely output-controllable iff the block matrix

$$[\mathbf{HG}\ \mathbf{HFG}\ \mathbf{HF}^2\mathbf{G}\ \ldots\ \mathbf{HF}^{n-1}\mathbf{G}]$$

is of rank n."

Output controllability is important because we can always measure $\mathbf{y}(t)$; if $\mathbf{x}(t)$ cannot be measured directly then it must be obtained by a knowledge of $\mathbf{y}(t)$. This leads us to the concept of observability.

Observability

A system is said to be completely observable if every change in the state of the system $\mathbf{x}(t)$ results in some change in the output of the system $\mathbf{y}(t)$. An unforced system (no control) can be described by the differential equations

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{F}\mathbf{x}(t) \\ \mathbf{y}(t) &= \mathbf{H}\mathbf{x}(t) \end{aligned} \quad (13)$$

Such a system is completely observable if for every t_0 and $t_f > t_0$, $\mathbf{x}(t_0)$ can be determined from the knowledge of $\mathbf{y}(t)$ for $t_0 \leq t \leq t_f$. The conditions for observability are stated in Theorem 3.

Theorem 3. "An unforced linear, finite-dimensional, continuous-time, constant dynamical system is completely observable iff the composite block matrix

$$[\mathbf{H}^T\ \mathbf{F}^T\mathbf{H}^T\ \ldots\ (\mathbf{F}^T)^{n-1}\mathbf{H}^T]$$

is of rank n."

The proof of Theorem 3 is also analogous to the proof of Theorem 1.

From Theorems 1 and 3, it can be seen that a linear system is completely controllable and completely observable if both matrices

$$[\mathbf{H}^T\ \mathbf{F}^T\mathbf{H}^T\ \ldots\ (\mathbf{F}^T)^{n-1}\mathbf{H}^T] \quad \text{and} \quad [\mathbf{G}\ \mathbf{FG}\ \mathbf{F}^2\mathbf{G}\ \ldots\ \mathbf{F}^{n-1}\mathbf{G}] \quad (14)$$

are of rank n.

In the modeling of a physical system, the control and measurement variables are known a priori. If an unobservable model is constructed for the system, the state vector will contain more state variables than can be determined from knowledge of the measurement variables. There will then exist a model of lower dimension which will correspond to the same input/output data but will be observable. Likewise, if the model is uncontrollable, the state variable vector is too large to be controlled by the input vector and there will exist another realization of smaller dimension.

Minimum Realizations and Algorithms

If a realization is completely observable and completely controllable then it is of minimum dimension and is called a minimum realization. This necessary and sufficient condition was presented and proven by Kalman (1963) and has been an important axiom in several algorithms devised to obtain realizations of minimum dimensions. The first two such algorithms were presented simultaneously by Kalman and by Gilbert (1963). Gilbert proposed a sequence of linear-algebra computations which resulted in the map: transfer function matrix → state-variable differential equations. In his algorithm, heavy emphasis was placed on the residue matrices of the poles of the transfer function matrix $\overline{\mathbf{Y}}(s)$. If s_i is a pole of $\overline{\mathbf{Y}}(s)$, then the residue matrix corresponding to that pole is defined to be

$$\mathbf{R}(i) = \lim_{s \to s_i} (s - s_i)\ \overline{\mathbf{Y}}(s)\,. \quad (15)$$

Gilbert also defined r_i to be the rank of $\mathbf{R}(i)$. The dimension of the minimum realization of $\overline{\mathbf{Y}}(s)$ was proven to be

$$\sum_{i=1}^{\text{no. of poles}} r_i = n$$

Since $\mathbf{R}(i)$ is of rank r_i, there are r_i independent columns in $\mathbf{R}(i)$ which can be combined to form the $p \times r_i$ matrix $\mathbf{H}(i)$. Then if we compute the $r_i \times m$ matrix $\mathbf{G}(i)$ by the relation

$$\mathbf{G}(i) = [\mathbf{H}(i)^T\mathbf{H}(i)]^{-1}\mathbf{H}(i)^T\mathbf{R}(i) \quad (16)$$

$\mathbf{R}(i)$ will equal $\mathbf{H}(i)\mathbf{G}(i)$. Gilbert showed that, if q were the number of poles, $\overline{\mathbf{Y}}(s)$ would have the irreducible realization

$$\mathbf{F} = \begin{bmatrix} s_1\mathbf{I}_{r_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & s_q\mathbf{I}_{r_q} \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}(1) \\ \vdots \\ \mathbf{G}(q) \end{bmatrix}$$

and $$\mathbf{H} = [\mathbf{H}(1)\ \ldots\ \mathbf{H}(q)]$$

where

$\mathbf{I}_{r_i}$ is an $r_i \times r_i$ unit matrix.

In Kalman's algorithm, a different approach was adopted. The construction of a minimum realization is divided into two steps: the formation of a controllable (or observable) realization and the subsequent reduction of this realization to minimum dimension. The essential ele-

ments of the procedure adopted by Kalman for the first step have been incorporated in the corresponding algorithms devised later by Wolovich and Falb (1969) and Mayne (1968) for the realization problem. Therefore, it will be explained here in some detail.

Suppose that the $p \times m$ transfer function matrix $\overline{\mathbf{Y}}(s)$ has elements $v_{ij}(s) = n_{ij}(s)/d_{ij}(s)$ which are relatively prime and assume that $d_{ij}(s)$ is of higher degree than $n_{ij}(s)$. The polynomials $n_{ij}(s)$ and $d_{ij}(s)$ are relatively prime if these is no root common to both. This restriction guarantees that there is no number that is both a zero and a pole of the quotient $n_{ij}(s)/d_{ij}(s)$. The algorithm first calculates the least common multiple of the denominator polynomials in each column of $\overline{\mathbf{Y}}(s)$ and calls it $g_j(s)$ for the jth column. Then h_j is defined as the degree of $g_j(s)$, and $n_1 \equiv \sum_{j=1}^{m} h_j$ with $p_k \equiv \sum_{j=1}^{k} h_j$. A new array of numerators is then defined as $n^*_{ij}(s) = n_{ij}(s)g_j(s)/d_{ij}(s)$. The polynomials $g_j(s)$ and $n^*_{ij}(s)$, will be of the form

$$\begin{aligned} g_j(s) &= s^{h_j} + \gamma_{j1}s^{h_j-1} + \ldots + \gamma_{jh_j} \\ n^*_{ij}(s) &= \alpha_{ij1}s^{h_j-1} + \alpha_{ij2}s^{h_j-2} + \ldots + \alpha_{ijh_j} \end{aligned} \tag{17}$$

because $g_j(s)$ is of higher degree than n^*_{ij}. Wolovich then formed the $n_1 \times m$ matrix $\mathbf{G}_c$ with zero elements except for $(\mathbf{G}_c)_{p_k,k} = 1$ and the $p \times n_1$ matrix $\mathbf{H}_c$ as

$$\mathbf{H}_c = \begin{bmatrix} \alpha_{11h_1} & \alpha_{11(h_1-1)} & \cdots & \alpha_{111} & \alpha_{12h_2} & \cdots & \alpha_{121} & \cdots & \alpha_{1m1} \\ \alpha_{21h_1} & \alpha_{21(h_1-1)} & \cdots & \alpha_{211} & \alpha_{22h_2} & \cdots & \alpha_{221} & \cdots & \alpha_{2m1} \\ \vdots & & & \vdots & \vdots & & & & \\ \alpha_{p1h_1} & \alpha_{p1(h_1-1)} & \cdots & \alpha_{p11} & \alpha_{p2h_2} & \cdots & \alpha_{p21} & \cdots & \alpha_{pm1} \end{bmatrix}$$

Then if

$$\mathbf{F}_c = \begin{bmatrix} \mathbf{F}_{c,1} & & \mathbf{0} \\ & \mathbf{F}_{c,2} & \\ \mathbf{0} & & \mathbf{F}_{c,m} \end{bmatrix}$$

where $\mathbf{F}_{c,j}$ is the comparison matrix corresponding to $g_j(s)$

$$\mathbf{F}_{c,j} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\gamma_{jh_j} & -\gamma_{jh_j-1} & & \cdots & -\gamma_{j1} \end{bmatrix}$$

Kalman proved that $\{\mathbf{F}_c, \mathbf{G}_c, \mathbf{H}_c\}$ will be a controllable realization of $\overline{\mathbf{Y}}(s)$. This realization can then be reduced to a minimum realization by finding its observable part.

The Kalman algorithm differs from those developed by Wolovich and Falb (1969) and Mayne (1968) essentially in the solution to the second step of the problem. Kalman's approach involves the formulation of some $\mathbf{T}$ such that

$$\mathbf{T}^T\mathbf{W}\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix}$$

where $\mathbf{W}$ is defined to be the observability matrix

$$\mathbf{W} = \int_0^1 (e^{-\mathbf{A}\tau})^T\mathbf{H}^T\mathbf{H}e^{-\mathbf{A}\tau}\, d\tau\,.$$

Then the observable part of the controllable realization, which was formed in the first part of the computation, is constructed by

$$\mathbf{T}^{-1}\mathbf{G}_c = \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}$$

$$\mathbf{H}_c\mathbf{T} = [\mathbf{0} \quad \mathbf{H}]$$

$$\mathbf{T}^{-1}\mathbf{F}_c\mathbf{T} = \begin{bmatrix} * & * \\ \mathbf{0} & \mathbf{F} \end{bmatrix}$$

The minimum realization is then $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$, where * represents extraneous elements.

In 1966, Ho and Kalman proposed another algorithm for minimum realization using modular theory. This method will be discussed in detail in the next section, since it is a most important procedure.

In 1968 Rosenbrock and Mayne independently introduced methods which arrived at realizations of minimum dimension by a reduction of higher order realizations. Mayne's algorithm utilized a step one procedure similar to that of Kalman. His second step was accomplished in the following way:

Given the realization $\{\mathbf{F}_c, \mathbf{G}_c, \mathbf{H}_c\}$ of dimension n construct the matrix $\mathbf{V}$ in the following manner. Let $\mathbf{h}_1, \ldots, \mathbf{h}_m$ denote the rows of $\mathbf{H}_c$. Then $\mathbf{h}_1 = \mathbf{v}_1$ and

$$\mathbf{V}^T = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_{n_k}]$$

where n_k is the rank of $[\mathbf{H}_c^T\ \mathbf{F}_c^T\mathbf{H}_c^T\ \mathbf{F}_c^{T2}\mathbf{H}_c^T \ldots \mathbf{F}_c^{Tn-1}\mathbf{H}_c^T]$. If $\mathbf{F}_c^T\mathbf{v}_j$ is linearly independent of $\mathbf{v}_1 \ldots \mathbf{v}_j$ then $\mathbf{v}_{j+1} = \mathbf{F}_c^T\mathbf{v}_j$. If not, and if $\mathbf{v}_j = (\mathbf{F}_c^T)^\xi\, \mathbf{h}_q$ for some ξ, then, $\mathbf{v}_{j+1} = \mathbf{h}_d$ where $\mathbf{h}_d$ is the first row of $\mathbf{h}_{q+1} \ldots \mathbf{h}_n$ which is linearly independent of $\mathbf{v}_1 \ldots, \mathbf{v}_j$. The process is continued until $i = n_k$, yielding the matrix $\mathbf{V}$. Then construct an $\mathbf{S}$ matrix such that

$$\mathbf{V}\,\mathbf{S} = \mathbf{I}_{n_k}$$

The realization

$$\{\mathbf{F} = \mathbf{V}\mathbf{F}_c\mathbf{S},\ \ \mathbf{G} = \mathbf{V}\mathbf{G}_c,\ \ \mathbf{H} = \mathbf{H}_c\mathbf{S}\}$$

will then be a minimum realization and the dimension will be n_k. The procedure for obtaining a minimum realization from any observable realization is analogous.

Rosenbrock's algorithm also relies on a two-step approach. The initial step, however, is the generation of a first-order matrix $\mathbf{P}(s)$ such that

$$\mathbf{P}(s) = \begin{bmatrix} s\mathbf{I}_n - \mathbf{F} & \mathbf{G} \\ -\mathbf{H} & \mathbf{D}(s) \end{bmatrix}$$

corresponding to the transform $\overline{\mathbf{Y}}(s)$ where $\mathbf{D}(s)$ is a matrix relating to $\overline{\mathbf{Y}}(s)$. The details of the construction of $\mathbf{P}(s)$ are in Rossen (1972) and will not be given here. The matrix $\mathbf{P}(s)$ is then reduced by row and column manipulations until a realization of minimum size is obtained.

Then in 1969, Wolovich and Falb developed yet another algorithm for the minimum realization problem. Kalman's solution to step one was incorporated in the algorithm to obtain a controllable realization. Then the matrix $\mathbf{K} = [\mathbf{G}, \mathbf{FG}, \ldots, \mathbf{F}^{n-1}\mathbf{G}]$ will be of rank n and we can form the matrix $\mathbf{L}$ from the first n linearly independent columns of $\mathbf{K}$ such that

$\mathbf{L} = [g_1, \mathbf{F}g_1, \ldots, \mathbf{F}^{\sigma_1-1}g_1, g_2, \ldots \mathbf{F}^{\sigma_2-1}g_2, \ldots, \mathbf{F}^{\sigma_m-1}g_m]$

where $g_1 \ldots g_m$ are the columns of $\mathbf{G}$. Wolovich and Falb then define $d_0 \equiv 0$ and $d_k \equiv \sum_{i=1}^{k} \sigma_i$ $k = 1, 2, \ldots m$ so that, if $\mathbf{l}_k^T$ is the d_kth row of $\mathbf{L}^{-1}$,

$$\mathbf{Q} = \begin{bmatrix} \mathbf{l}_1^T \\ \mathbf{l}_1^T\mathbf{F} \\ \vdots \\ \mathbf{l}_1^T\mathbf{F}^{\sigma_1-1} \\ \vdots \\ \mathbf{l}_m^T\mathbf{F}^{\sigma_n-1} \end{bmatrix}$$

This matrix **Q**, a Lyapunov transformation, can be used precisely as **T** was used in the Kalman algorithm to construct the observable part of the controllable realization.

In 1971 Silverman published an interesting summary of some important aspects of the Ho-Kalman algorithm and the Tether (1970) algorithm (to be discussed shortly). The theory was also extended to the analysis of time-varying systems.

Computer Algorithms

Recently, computer programs, utilizing these algorithms in the construction of minimum realizations, have appeared in the literature. In his manual for the ASP matrix manipulation program, Kalman (1966) included a chapter on the use of the computer program to obtain minimum realizations by Ho's algorithm. Kalman outlined the steps for obtaining a solution and then solved several sample problems. The listing of the program is included in the discussion. Then in 1968 the theory derived by Wolovich and Falb was incorporated into a computer program for the construction of a minimum realization $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$ from the system's transfer function. This program was the work of Wolovich and Bonna (1970) and provides a very thorough documentation of the algorithm. Block diagrams are presented for each of the subroutines used in the program and a complete discussion is given for the operation of the algorithm. In 1969, Roberts published a discussion of the computer program which he derived based on the theory presented by Mayne. The program was developed for the IBM 7094 computer. The overall structure of the program is given and the subroutines required are mentioned, but the flow diagrams or listings are not available in the paper.

THE ALGORITHM OF HO AND KALMAN

As an aid in understanding the theory of minimum realizations, we will now examine in some detail the algorithm presented by Ho and Kalman (1966). This is one of the simplest and most useful algorithms that have been proposed for constructing realizations of minimum dimension. The method assumes that

1. the system is linear
2. the system has constant coefficients
3. that (1) or (2) are valid equation forms
4. the system is noise-free and
5. the initial state of the system is $\mathbf{x}(0) = \mathbf{0}$.

The impulse response data can be given in the time- or the s-domain since the use of Markov parameters allows a direct correlation between the two domains.

The problem can be stated as:

"Given a sequence of $p \times m$ constant matrices, $\mathbf{Y}_k$, $k = 0, 1, 2, \ldots$, find a triple $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$ of constant matrices such that

$$\mathbf{Y}_k = \mathbf{H}\mathbf{F}^k\mathbf{G} \qquad k = 0, 1, 2, \ldots \text{"}$$

The solution to this problem follows from:

"This problem will have a solution of finite dimension if there exists an r and constants α_i such that

$$\mathbf{Y}_{r+j} = \sum_{i=1}^{\infty} \alpha_i \mathbf{Y}_{r+j-i} \quad \text{for all } j \geq 0 \qquad (18)$$

where r is the degree of the annihilating polynomial of $\mathbf{F}_{\min}$ (see below)."

The necessity of (18) is a consequence of the existence of such a polynomial for a finite dimensional matrix. Given a finite dimensional $\mathbf{F}$, there exists a $\Psi(\mathbf{F})$ such that

$$\Psi(\mathbf{F}) = \sum_{i=0}^{q} \beta_i \mathbf{F}^{q-i} = \mathbf{0} \qquad (19)$$

Since $\mathbf{Y}_k = \mathbf{H}\mathbf{F}^k\mathbf{G}$, it follows that

$$\mathbf{0} = \mathbf{H}\mathbf{F}^j\Psi(\mathbf{F})\mathbf{G} = \sum_{i=0}^{q} \beta_i \mathbf{H}\mathbf{F}^{q-i+j}\mathbf{G} \qquad (20)$$

and (18) holds for $r = q$ and $\alpha_i = -\beta_i/\beta_0$. The sufficiency of (18) follows in the solution of the problem. The proof will show that a realization exists by using (18) to find such a realization.

Ho assumes that the value of r is known. If the impulse-response function is given in the s-domain, the value of r is determined easily from the poles of the elements of the transform matrix (Kalman, 1964). For a rational matrix $\overline{\mathbf{Y}}(s)$, the value of r is simply the degree of the least common polynomial denominator of $\overline{\mathbf{Y}}(s)$. $\overline{\mathbf{Y}}(s)$ is said to be a rational matrix if each element $\overline{Y}_{ij}(s)$ can be expressed as a ratio of polynomials in s. If the input/output description is obtained in the time domain as experimental data, numerical methods or curve fitting techniques must be used to determine r. Extensions of Ho's algorithm deal with this problem in greater detail (Tether, 1970).

Ho's algorithm centers on "the generalized Hankel matrix" an $r \times r$ block matrix composed of the Markov parameters.

$$\mathbf{S}_r = \begin{bmatrix} \mathbf{Y}_0 & \mathbf{Y}_1 & \cdots & \mathbf{Y}_{r-1} \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_r \\ \vdots & \vdots & & \vdots \\ \mathbf{Y}_{r-1} & \mathbf{Y}_r & \cdots & \mathbf{Y}_{2r-2} \end{bmatrix} = [\mathbf{Y}_{i+j-2}]$$

The rows and columns of the Hankel matrix are shifted to define a set of matrices $\tau^l \mathbf{S}_r$ such that

$$\tau^l \mathbf{S}_r = [\mathbf{Y}_{i+j+l-2}] \qquad l = 0, 1, 2, \ldots$$

The rank of $\mathbf{S}_r$, s, is equal to the dimension n_0 of the minimum realization of the system. To confirm this point, consider a minimum realization $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$. Then form $\mathbf{S}_r = \mathbf{V}^T\mathbf{W}$ where

$$\mathbf{V} = [\mathbf{H}^T \; \mathbf{F}\mathbf{H}^T \ldots (\mathbf{F}^T)^{r-1}\mathbf{H}^T]$$
$$\mathbf{W} = [\mathbf{G} \; \mathbf{F}\mathbf{G} \ldots \mathbf{F}^{r-1}\mathbf{G}] \qquad (21)$$

Since $\mathbf{F}$ is $(n_0 \times n_0)$ we can see that rank $\mathbf{V} \leq n_0$ and rank $\mathbf{W} \leq n_0$. Therefore rank $\mathbf{S}_r \leq n_0$. But if rank $\mathbf{S}_r < n_0$, Ho's algorithm would give rise to a realization of dimension less than n_0. However, since $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$ is assumed to be of minimum dimension, a contradiction develops and rank $\mathbf{S}_r = n_0$. Thus, if the algorithm does indeed construct a realization of dimension s, that dimension is minimal.

To show that a realization of dimension s can be constructed and to give the steps necessary for that construction remains. After the formation of the block matrix $\mathbf{S}_r$, matrices $\mathbf{P}$ and $\mathbf{Q}$ are found such that

$$\mathbf{P}\mathbf{S}_r\mathbf{Q} = \begin{bmatrix} \mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{J} = \mathbf{U}_s^T\mathbf{U}_s \qquad (22)$$

where $\mathbf{I}_s$ is an $s \times s$ unit matrix and $\mathbf{U}_s = [\mathbf{I}_s \; \mathbf{0}]$. This can be accomplished by standard linear algebra techniques. The matrix $\mathbf{E}_p$ is then defined to be the $1 \times r$ block matrix $[\mathbf{I}_p \; \mathbf{0}_p \; \mathbf{0}_p \ldots \mathbf{0}_p]$. A similar definition is made for $\mathbf{E}_s$. The realization can then be constructed by use of the following theorem.

Theorem 4. "Whenever a finite-dimensional realization exists, a realization of dimension s can be constructed by

forming

$$\mathbf{F}_s = \mathbf{U}_s[\mathbf{JP}(\tau\mathbf{S}_r)\mathbf{QJ}]\mathbf{U}_s^T$$
$$\mathbf{G} = \mathbf{U}_s[\mathbf{JPS}_r\mathbf{E}_m^T] \qquad (23)$$
$$\mathbf{H} = [\mathbf{E}_P\mathbf{S}_r\mathbf{QJ}]\mathbf{U}_s^T$$

where the block matrices are computed as shown above."

The verification of Theorem 4 can be found in Ho and Kalman (1966). The preceding proof indicates that the realization will be minimal.

The Ho algorithm is one method of obtaining an internal description of a linear system from its impulse-response function. Several other methods involve different approaches to the minimum realization problem, but the overall influence of controllability and observability, that is (23), is always present. The Ho algorithm is one of the simplest methods of computing a realization and also has the advantage of being easily adapted to the proof of some interesting theoretical properties of minimum realizations.

EXACT LINEAR SYSTEM REALIZATIONS

We have already mentioned that a given linear system can be represented by an infinite number of triples {**F**, **G**, **H**}. It is also true that there are an infinite number of realizations of the system which are of minimum dimension. These realizations are then equivalent and any one of them will suffice to define the system.

Equivalent Minimum Realizations

Stated in another fashion we consider two minimal realizations $\{\mathbf{F}_1, \mathbf{G}_1, \mathbf{H}_1\}$ and $\{\mathbf{F}_2, \mathbf{G}_2, \mathbf{H}_2\}$ of the same $\mathbf{Y}_k$. These two realizations must be isomorphic. That is to say that there exists a nonsingular matrix **T** such that

$$\mathbf{F}_2 = \mathbf{T}\,\mathbf{F}_1\mathbf{T}^{-1}$$
$$\mathbf{G}_2 = \mathbf{T}\,\mathbf{G}_1 \qquad (24)$$
$$\mathbf{H}_2 = \mathbf{H}_1\mathbf{T}^{-1}$$

If **T** is nonsingular and of dimension n_0, $\mathbf{I} = \mathbf{T}^{-1}\mathbf{T}$. Thus

$$\mathbf{H}_2\mathbf{F}_2^k\mathbf{G}_2 = \mathbf{H}_1\mathbf{T}^{-1}\,(\mathbf{T}\mathbf{F}_1^k\mathbf{T}^{-1})^k\mathbf{T}\mathbf{G}_1$$
$$= \mathbf{H}_1\mathbf{F}_1^k\mathbf{G}_1 \qquad (25)$$

Diagonal Minimum Realizations

Any minimum realization can be obtained from any other minimum realization through the choice of an appropriate **T**. It would be of special interest if a minimum realization could be found such that $\mathbf{F}_2$ would be a diagonal matrix. Premultiplication by $\mathbf{T}^{-1}$ in (24) would then yield

$$\mathbf{F}_1\mathbf{T}^{-1} = \mathbf{T}^{-1}\mathbf{F}_2 = \mathbf{T}^{-1}\mathbf{D} \qquad (26)$$

where **D** is a diagonal minimal realization. Such a **D** exists whenever $\mathbf{F}_1$ is nonsingular and has distinct eigenvalues and the diagonal of **D** is formed by the eigenvalues of $\mathbf{F}_1$. $\mathbf{T}^{-1}$ is merely the matrix of eigenvectors of $\mathbf{F}_1$. Thus the triple $\{\mathbf{D}, \mathbf{G}_2, \mathbf{H}_2\}$ can be quickly obtained from any minimum realization {**F**, **G**, **H**} using linear algebraic techniques. The eigenvalues can be found analytically by solving the characteristic equation of $\mathbf{F}_1$ and the corresponding eigenvectors can be found by solving (26) for $\mathbf{T}^{-1}$. The eigenvectors will not be unique but can be found if some additional assumption is made about each vector (that is, $\sum_i (T_{ij}^{-1})^2 = 1$ or $T_{1j}^{-1} = 1$). It should also be noted that all minimum realizations have the same set of eigenvalues and that these eigenvalues are parameters of the system itself. A system can only have one such diagonal **F** (or **D**) matrix.

The solution of the linear O.D.E.'s when **F** is the diagonal matrix **D** is [see (6) and (7)]

$$\mathbf{y}(t) = \mathbf{H}\int_0^t \exp\,[\mathbf{D}(t-\tau)]\,\mathbf{G}\mathbf{u}(\tau)\,d\tau \qquad (27)$$

and the impulse response function matrix becomes

$$\mathbf{Y}(t) = \mathbf{H}\exp\,[\mathbf{D}t]\,\mathbf{G} \qquad (28)$$

The necessary and sufficient condition for an asymptotically stable impulse response is that the eigenvalues of the **F** matrix must all be negative. This condition ensures that the impulse response will damp out to zero as $t \to \infty$. This also ensures that the transient behavior of other input function responses will damp out with time.

Assume that the **D** matrix has diagonal elements, λ_i, $i = 1, 2, \ldots n$, such that each λ_i is negative. Assume also that at least one eigenvalue λ_j is much larger in absolute magnitude than the smallest eigenvalue $|\lambda_i|_{\min}$. If there is a wide range in the magnitudes of the eigenvalues of a matrix, the matrix is said to be stiff. Such matrices can be difficult to handle numerically. Calculation of determinant, minors, and inverses can result in large truncation errors. If the larger eigenvalues could be discarded, it would lead to a system of smaller dimension which could be treated more easily, would require less computation time and storage space, and would not have the truncation drawbacks of stiff systems. The essential question, then, is whether the simplification of the system will significantly decrease the accuracy of response predictions made from the realization, thereby negating the model's usefulness as an internal description of the system.

Davison (1966) has made some studies in this area and concluded that the dimension of the system could be reduced significantly by this technique. The simplified realization, of course, no longer models the original system exactly. In fact, as $k \to \infty$, it is apparent that the $\mathbf{Y}_k$ become dominated by the largest eigenvalues so that the elimination of these eigenvalues will drastically change the $\mathbf{Y}_k$. The resulting sequence of $\mathbf{Y}_k$ will, however, model the system's impulse and step responses very well at large times because the $e^{-\lambda_i t}$ terms, when λ_i is very large and negative, become negligible at such large times. In the step response of the realization, there will be a small error in the steady state values, that is, the simplified system will not have the same steady state values as the original system. Davison (1968) deals with this problem by defining a new set of state variables $\mathbf{x}_1 = \mathbf{D}\mathbf{x}$ where **D** is a diagonal matrix picked so that the steady state response is identical to that of the original system. He then demonstrates the technique by reducing a ninth-order system to a fourth-order system with a negligible loss of accuracy.

The technique retains the dominant modes (that is, eigenvalues closest to zero) of the original system and they respond in the same proportion for given forcing function inputs. The simplified system model thus reacts in the same general way that the original system reacts. The main discrepancy occurs at small times when the large eigenvalues have their greatest effect. If, for example, one is interested in the initial slope of the step response, the simplification technique would be undesirable because the value of $\mathbf{Y}_0$ can be altered significantly by the simplifications (recall that $\mathbf{Y}_0$ is the initial value of the impulse response function). It is important to point out that, although the terms involving the large negative eigenvalues will die out fastest at large times, they may be the most important terms at very small times. The procedure by Davison ignores possible deviations around

$t = 0$ in an attempt to get a good overall fit of the system response.

Connection to Moment Analysis

The minimum realization algorithm presented by Ho and Kalman provided a method of obtaining a realization of a system from its pulse response function in terms of Markov parameters. These parameters can be difficult to measure when the data is in the form of an experimental time plot of the impulse response. The Markov parameters, coefficients in a negative power series expansion of $\overline{\mathbf{Y}}(s)$ in terms of s^{-k}, can be interpreted as time derivatives of the impulse response matrix. Bruni et al. (1968) suggested than an expansion of $\overline{\mathbf{Y}}(s)$ in positive powers of s be used to construct a realization. The coefficients in this expansion would be the matrices $\mathbf{HF}^{-k}\mathbf{G}$ and these parameters could be treated in the Ho algorithm to get a triple of matrices $\{\mathbf{F}^{-1}, \mathbf{G}, \mathbf{H}\}$, from which the actual realization could be obtained easily. The real advantage of this change would be in the case for which the external description of the system is in the form of an empirical time-response curve because the $\mathbf{HF}^{-k}\mathbf{G}$ correspond to the moments of the impulse response. To show this we rewrite (6)

$$\mathbf{Y}(t) = \mathbf{H} \exp [\mathbf{F}t]\, \mathbf{G} \tag{29}$$

for the impulse response matrix $\mathbf{Y}(t)$ and define the moments of this response by

$$\mu_k = \int_0^\infty t^k \mathbf{Y}(t)\, dt \qquad k = 0, 1, 2, \ldots \tag{30}$$

Substituting (29) into (30), carrying out the integration by parts k times and assuming the system is asymptotically stable, that is,

$$\lim_{t \to \infty} \exp [\mathbf{F}t] = \mathbf{0} \tag{31}$$

the following relations are found

$$\mu_k = (-1)^{k+1} k!\ \mathbf{HF}^{-k-1}\mathbf{G} \qquad k = 0, 1, 2, \ldots \tag{32}$$

Thus

$$\begin{aligned} \mu_1 &= \mathbf{HF}^{-2}\mathbf{G} \\ \mu_2 &= 2\mathbf{HF}^{-3}\mathbf{G} \\ &\cdot \\ &\cdot \\ &\cdot \\ \mu_n &= n\ \mathbf{HF}^{-(n+1)}\mathbf{G} \end{aligned} \tag{33}$$

connect the impulse moments with the triple $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$. If we further define

$$\mu_k^* = \frac{(-1)^k}{(k-1)!}\, \mu_{k-1} \qquad k = 1, 2, \ldots$$

and

$$\mu_0^* = \mathbf{Y}(t)|_{t=0} = \lim_{s \to \infty} s\overline{\mathbf{Y}}(s)$$

then (30) can be written as

$$\mu_k^* = \mathbf{HF}^{-k}\mathbf{G} \qquad k = 0, 1, 2, \ldots \tag{34}$$

In the presence of noise, the computation of moments would be preferable to the determination of local time derivatives or the determination of a power series fit to the time curve. Bruni and his coworkers proven that the solution by this method is indeed a minimum realization.

The calculation of the true realization from the triple $\{\mathbf{F}^{-1}, \mathbf{G}, \mathbf{H}\}$ involves an inversion of the $\mathbf{F}^{-1}$ matrix and, of course, an assumption that $\mathbf{F}$ is nonsingular so that $\mathbf{F}^{-1}$ does exist. This is always a valid assumption when the impulse response $\mathbf{Y}(t)$ is asymptotically stable and if $\mathbf{F}$ is minimal. The Bruni modification cannot be used if $\mathbf{F}$ is singular because the moments would then not be finite and the treatment of the matrices $\mathbf{HF}^{-k}\mathbf{G}$ would not be possible. The applicability of the approach can be determined by an examination of the asymptotic stability of the pulse response. In the case where Bruni's adaptation cannot be used, the Ho algorithm can still be applied to the Markov parameters $\mathbf{HF}^{-k}\mathbf{G}$ to obtain a realization.

It is important to note that when the pulse-response is asymptotically stable, a combination of the Ho approach and the Bruni approach might be employed and indeed may prove to be of significant value. For example, the parameters

$$\mathbf{HF}^{-1}\mathbf{G},\ \mathbf{HG},\ \mathbf{HFG},\ \mathbf{HF}^2\mathbf{G},\ \ldots$$

might be used to construct a realization. Ho's method is applicable for any sequence of $2r$ parameters of the form $\mathbf{HF}^k\mathbf{G}$ where $k = \ldots -2, -1, 0, 1, 2, \ldots$. In the example sequence shown above, the realization determined from Ho's algorithm would be $\{\mathbf{F}, \mathbf{G}, \mathbf{HF}^{-1}\}$ which can easily be converted to the true realization $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$. The $\mathbf{Y}_k$ where $k \geq 0$ would assure an accurate fit of the system in the neighborhood of the origin while the $\mathbf{Y}_k$ where $k < 0$ would improve the overall predictions of future behavior and the predictions of new steady state operating conditions.

SUBOPTIMAL REALIZATIONS

The algorithms that have been proposed for finding minimum dimension realizations have different degrees of complexity but tend to require a considerable amount of cumbersome mathematical manipulation. It is therefore of some value to investigate simple methods of constructing suboptimal solutions to the realization problem. Such a solution might require significantly less work while still yielding a realization to the system which could, if necessary, be reduced to minimum size by the methods suggested by Mayne (1968) and Rosenbrock (1968). Kalman (1963), in his treatment discussed in the previous section, presented a method of obtaining a realization of a transfer function $\overline{\mathbf{Y}}(s)$ where the dimension of the realization is equal to

$$N_0 = \min \left\{ \sum_{i=1}^{p} \alpha_i \quad \sum_{j=1}^{m} \beta_j \right\} \tag{35}$$

where α_i and β_j are the number of distinct poles in the ith row and jth column, respectively. N_0 is an upper bound for n_0, the dimension of the minimum realization. Roveda and Schmid (1970) recently proposed another very simple method for calculating a suboptimal solution of dimension

$$N_0^* = \sum_{k=1}^{q} r_k^* \tag{36}$$

where q is the number of distinct poles of $\overline{\mathbf{Y}}(s)$, and r_k^* is the minimum number of rows and columns of the residue matrix of the kth pole, ζ_k, of $\overline{\mathbf{Y}}(s)$ that contain all of the nonzero elements of that residue matrix. The residue matrix of the kth pole of $\overline{\mathbf{Y}}(s)$ is obtained by taking the limit as s approaches ζ_k of $(s - \zeta_k)\overline{\mathbf{Y}}(s)$. It can be shown that $n_0 \leq N_0^* \leq N_0$ so that N_0^* is more restrictive than N_0 as an upper bound on N_0. N_0^* has been shown to be the minimum dimension for realizations having an invariant structure with respect to the coefficients of the elements $\overline{\mathbf{Y}}_{ij}(s)$. That is, if the numerical coefficients were altered randomly, the N_0^* would not change. It is the

least upper bound on the dimension of minimum realization of all transfer functions with the distribution of poles found in $\overline{\mathbf{Y}}(s)$. The algorithms for constructing a suboptimal solution to the realization problem are less cumbersome than those used in the construction of a minimum realization. Roveda and Schmid presented an algorithm by which the matrices could be obtained by inspection from the residue matrices of $\overline{\mathbf{Y}}(s)$ with respect to the poles of $\overline{\mathbf{Y}}(s)$. Suboptimal methods of this type are useful in obtaining a quick and easy solution to the realization problem.

MINIMUM PARTIAL REALIZATIONS

The minimum realization algorithms discussed previously deal with linear, finite-dimensional, constant-dynamical systems for which either the transfer function or the sequence of Markov parameters is known exactly. Suppose, however, that only partial information about the system is available. One method of approach to the modeling of such a system involves the construction of a transfer function which provides a reasonable fit to the available input-output data. In this technique, the form of the transfer function is chosen a priori and the constants in each element of the transfer function matrix are determined by parameter estimation (Kallina, 1970). Any of the minimum realization algorithms can then be used to obtain a realization for this transfer function. This method can be useful but suffers from the need to a priori fix the form of the matrix $\overline{\mathbf{Y}}(s)$ and thus restricts the flexibility of the model. It is, then, desirable to have a method for finding an approximate realization to the system which could be obtained directly from experimental data in the same way that the Ho and Kalman algorithm is used to find an exact realization for completely specified systems.

The Ho-Kalman algorithm requires that $2r$ Markov parameters or $2r$ moment matrices be specified exactly. Consider, however, a system which is not finite-dimensional but rather is a distributed parameter system. In this case, any finite sequence of parameters $\{\mathbf{Y}_0, \mathbf{Y}_1, \ldots \mathbf{Y}_{N_0}\}$ would provide insufficient data for the solution to the problem. Similarly it may be that the system is finite-dimensional but too few parameters are available for an exact Ho solution. In addition, it is likely that there will be some measurement error in the $\mathbf{Y}_k$ which have been obtained empirically. Such errors lead to difficulties in the determination of r, the degree of the annihilating polynomial of $\mathbf{F}_{\min}$. Also, the operation of the algorithm would be hindered because the rows and columns in the Hankel matrix would not cancel exactly as they would do in the noise-free, completely accurate case. In these frequently encountered instances, the Ho algorithm by itself is not sufficient to obtain a realization for the system.

With these thoughts in hand we now consider a sequence of $p \times m$ matrices $\mathbf{Y}_k$, $k = 0, 1, 2, \ldots, N_0 - 1$. We then ask if it is possible to construct a realization which will have the identical initial N_0 Markov parameters $\mathbf{Y}_k$. Further, if such a realization can be constructed, we ask how this construction can be accomplished. The answer to these questions resides in a method called the minimum partial realization algorithm which is most useful in modeling systems for which exact linear models cannot be obtained from experimental data. As such, the algorithm is extremely important.

Tether's Minimum Partial Realization Algorithm

Solutions to the questions raised above were proposed by Tether (1970) and by Kalman (1970). We will examine the algorithm presented by Tether in some detail, but we will need to clarify the basic concepts of partial realizations and minimal partial realizations. The triple $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$ is said to be a partial realization of order N_0 of $\mathbf{Y}_k$ if $\mathbf{Y}_k = \mathbf{H}\mathbf{F}^k\mathbf{G}$ for $k = 0, 1, \ldots, N_0 - 1$. In addition, $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$ is said to be a minimum partial realization of order N_0 iff the dimension of $\mathbf{F}$ is minimal among all other partial realizations as defined above. In the modeling of continuous-time systems, the partial realization concept is equivalent to matching the first N_0 terms of the power series expansion of $\overline{\mathbf{Y}}(s)$ or $\mathbf{Y}(t)$ as shown in (10) and (11).

There always exists a partial realization which fits the given sequence of Markov parameters. This is sufficiently important that we state this feature in concise terms, namely:

> "Every finite sequence of N_0 $p \times m$ constant matrices $\{\mathbf{Y}_0, \mathbf{Y}_1, \mathbf{Y}_{N_0-1}\}$ admits an extension sequence $\{\mathbf{Y}_{N_0}\ \mathbf{Y}_{N_0+1} \ldots\}$ for which a completely controllable and completely observable partial realization $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$ of order N_0 exists via Ho's algorithm."

The proof of this statement can be found in Tether (1970) and asserts the existence of a partial realization.

As in the Ho algorithm, we can restate the problem in terms of the Markov parameters. Then, the statement of the minimum partial realization problem becomes

> "Given a finite sequence $\{\mathbf{Y}_0, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{N_0-1}\}$ of constant $p \times m$ matrices with real elements, find a triple of constant matrices $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$ such that
>
> $$\mathbf{Y}_k = \mathbf{H}\mathbf{F}^k\mathbf{G} \qquad k = 0, 1, 2, \ldots, N_0 - 1$$
>
> and $\mathbf{F}$ is of minimum dimension."

We should say something about the uniqueness of the minimum partial realization because if this realization is unique, then it is a true minimum realization in the Ho-Kalman sense. A minimum realization is said to be unique if all other minimum realizations can be obtained from the given realization by a transformation involving some nonsingular matrix $\mathbf{T}$ in the manner described earlier. A minimal partial realization will be unique (in the same sense) if the extension sequence defined by $\mathbf{Y}_k = \mathbf{H}\mathbf{F}^k\mathbf{G}$, $k = N_0, \ldots$ is unique.

The central theme of the Tether algorithm for solving the above stated problem will shortly be given as Theorem 5. Before giving this theorem however we need to introduce some notation and specify certain concepts. Thus we define a notational change for the Hankel matrix

$$\mathbf{H}_{N',N} = \begin{bmatrix} \mathbf{Y}_0 & \cdots & \mathbf{Y}_{N-1} \\ \vdots & & \\ \mathbf{Y}_{N'-1} & \cdots & \mathbf{Y}_{N'+N-2} \end{bmatrix} \tag{37}$$

and let $\rho\mathbf{H}_{N',N} = \text{rank } (\mathbf{H}_{N',N})$. Now we can state the following:

> "Given a finite sequence of $p \times m$ constant matrices $\{\mathbf{Y}_k\}$ $k = 0, 1, 2, \ldots, N_0 - 1$ satisfying
>
> $$\rho\mathbf{H}_{N',N} = \rho\mathbf{H}_{N'+1,N} = \rho\mathbf{H}_{N',N+1}$$
>
> for some N',N such that $N' + N = N_0$, the extension of the sequence $\{\mathbf{Y}_0 \ldots \mathbf{Y}_{N_0-1}\}$ to $\{\mathbf{Y}_0 \ldots \mathbf{Y}_{N_0-1}\ \mathbf{Y}_{N_0} \ldots\}$ for which $\rho\mathbf{H}_{m',m} = \rho\mathbf{H}_{N',N}$ where $m' + m = N_0 + k$, $k = 0, 1, 2, \ldots$, is unique."

This statement is proven in Tether (1970) and leads di-

rectly to the criterion for the existence of a unique minimum partial realization, namely,

> "Let $\{\mathbf{Y}_0 \ldots \mathbf{Y}_{N_0-1}\}$ be an arbitrary finite sequence of $p \times m$ real matrices and let $\mathbf{H}_{i,j}$, $i + j \leq N_0$, be a corresponding Hankel matrix. Then, a minimum partial realization given by Ho's algorithm is unique and realizes the sequence up to and including the N_0-th term iff there exists positive integers N' and N such that
>
> (a) $N' + N = N_0$
>
> (b) $\rho\mathbf{H}_{N',N} = \rho\mathbf{H}_{N'+1,N} = \rho\mathbf{H}_{N',N+1}$."

This criterion is nothing more than a direct consequence of Ho's algorithm. We can employ this algorithm (Ho's) with

$$n_0 = \rho\mathbf{H}_{N',N},$$

using $\mathbf{H}_{N',N}$ as defined in (37) and defining $_T\mathbf{H}_{N',N}$ as before. The resulting minimal partial realization is unique because the extension sequence $\mathbf{Y}_k = \mathbf{HF}^k\mathbf{G}$ $k = N_0, N_0 + 1, \ldots$ generated by the realization satisfies

$$\rho\mathbf{H}_{N'+i,\,N+j} = \rho\mathbf{H}_{N',N}$$

for all $i, j \geq 0$. Thus by the criterion above the extension sequence is unique and therefore the partial realization is unique.

If (a) and (b) are not satisfied, Ho's algorithm requires that new matrices $\{\mathbf{Y}_{N_0}, \ldots \mathbf{Y}_{P_0-1}\}$ must be satisfied until $\rho\mathbf{H}_{N',N} = \rho\mathbf{H}_{N'+1,N} = \rho\mathbf{H}_{N',N+1}$ where $N' + N = P_0$. These matrices may be completely or partially arbitrary and therefore since $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ are functions of $\{\mathbf{Y}_{N_0}, \ldots, \mathbf{Y}_{P_0-1}\}$ they may not be unique.

The minimum dimension for a partial realization of the system in the above criterion is obviously

$$n_0 = \rho\mathbf{H}_{N',N} \tag{38}$$

If the above criterion is not satisfied the lower bound on the minimum dimension is the rank of the matrix

$$\begin{bmatrix} \mathbf{Y}_0 & \mathbf{Y}_1 & \mathbf{Y}_2 & \cdots & \mathbf{Y}_{N_0-2} & \mathbf{Y}_{N_0-1} \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \mathbf{Y}_3 & \cdots & \mathbf{Y}_{N_0-1} & * \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \mathbf{Y}_{N_0-2} & \mathbf{Y}_{N_0-1} & * & \cdots & * & * \\ \mathbf{Y}_{N_0-1} & * & * & \cdots & * & * \end{bmatrix}$$

(where * signifies a blank to be filled in at a later time) because the rank cannot decrease as the * are filled. The minimum dimension can be found by counting the number of linearly independent rows or by counting the number of linearly independent columns. Tether showed that the minimum dimension could then be computed with the aid of the following statement which he proved.

> "Let $\{\mathbf{Y}_0\ \mathbf{Y}_1 \ldots \mathbf{Y}_{N_0-1}\}$ be a finite sequence of $p \times m$ constant matrices with real elements and let $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$ be a partial realization whose first N_0 Markov parameters are equal to the given sequence. Then the dimension of the minimum partial realization $n_{\min}$ satisfies the following inequality,

$$n_{\min} \geq n(N_0) \equiv \sum_{j=1}^{N_0} \rho\mathbf{H}_{j,N_0+1-j} - \sum_{j=1}^{N_0} \rho\mathbf{H}_{j,N_0-j}." \tag{39}$$

Thus, $n(N_0)$ is the lower bound for the dimension of the partial realization for any sequence of Markov parameters. But does a realization of the sequence exist such that its dimension is $n(N_0)$?

Let $N'(N_0)$ be defined to be the smallest integer such that every row of the block row $[\mathbf{Y}_{N'(N_0)} \ldots \mathbf{Y}_{N_0-1}]$ is linearly dependent on the rows of the Hankel matrix $\mathbf{H}_{N'(N_0),N_0-N'(N_0)}$. Then, let $N(N_0)$ be the first integer so that the columns of $[\mathbf{Y}_{N(N_0)} \ldots \mathbf{Y}_{N_0-1}]$ are linearly dependent on the columns of $\mathbf{H}_{N_0-N(N_0),N(N_0)}$. It has been shown that any extension of the sequence whose realization is of dimension $n(N_0)$ satisfies

$$\begin{aligned} \rho\mathbf{H}_{N'(N_0),N(N_0)} &= \rho\mathbf{H}_{N'(N_0)+1,N(N_0)} \\ &= \rho\mathbf{H}_{N'(N_0),N(N_0)+1} \end{aligned} \tag{40}$$

for the extension.

It should be noted that $N'(N_0)$ and $N(N_0)$ are invariants of the sequence of $\mathbf{Y}_k$ and are related to the controllability and observability of the system realization. The realization therefore has the property that

$$\rho[\mathbf{G}\ \mathbf{FG}\ \mathbf{F}^2\mathbf{G} \ldots \mathbf{F}^{N-1}\mathbf{G}] = n(N_0)$$

and

$$\rho[\mathbf{H}\ \mathbf{HF}\ \mathbf{HF}^2 \ldots \mathbf{HF}^{N'-1}]^T = n(N_0) \tag{41}$$

The groundwork is now set for the central theorem in Tether's algorithm.

Theorem 5. "Let $\{\mathbf{Y} \ldots \mathbf{Y}_{N_0-1}\}$ be a fixed partial sequence of $p \times m$ constant matrices with real coefficients and let $n(N_0)$, $N(N_0)$, $N'(N_0)$ be the integers defined above. Then

(a) $n(N_0)$ is the dimension of the minimum partial realization

(b) $N(N_0)$ and $N'(N_0)$ are the smallest integers such that (40) holds simultaneously for all minimum extensions

(c) There is a minimal extension of order $P(N_0) = N(N_0) + N'(N_0)$ for which $n(N_0)$ is the dimension of the realization computed by Ho's algorithm but which is in general not unique

(d) Every extension to $P(N_0)$ is uniquely determined thereafter."

The proof of (a), (b), and (d) follows readily from the material we have already stated if (c) is true. Tether indicates that (c) is indeed true by outlining a method for the construction of such a realization. We will not give the details of the method here. The reader is referred to Tether (1970) for a detailed discussion.

The partial realization method, then can be used to construct approximate state-space representations of physical systems when only partial input/output information is available. In the construction of models based on experimental data, the accuracy of the measured moments and/or Markov parameters is often limited by the presence of noise and the inherent measurement errors in the recording of the impulse response. The Ho algorithm will not be able to construct an exact, finite dimensional realization for systems with measurement errors. The Tether algorithm is useful for such systems because it can generate an approximate model which agrees with the available data.

Comparison With Other Identification Methods

We have shown that the theory of minimum realizations can be useful in obtaining a description of the internal workings of a linear system. Other methods for accomplishing this same result have also been proposed and it will be helpful to place the minimum realization in proper perspective in relation to these other methods. In particular, the two most frequently used techniques are moment and frequency response analysis, (see Kropholler, 1970; Gibilaro and Lees, 1969; Buffham and Gibilaro, 1970).

If all systems were truly linear, constant-coefficient, and finite-dimensional and if all measurements could be made to infinite accuracy, the minimum realization algorithms would be superior to the frequency response methods and moment methods for the identification of unknown systems. The minimum realization solution is an exact solution while the others are approximations to the system. In real systems, however, measurements are not exact and, in some cases, only a finite number of moments or Markov parameters are available. If this number is insufficient for a Ho type of solution, then an exact solution would be impossible. How would the different types of solutions compare in this case?

Let us assume that we have a linear, finite-dimensional, constant dynamical system and that only a few moments are known and that these moments may contain measurement errors. Given a fixed number of exact moments, the minimum realization method can extract more information, by simultaneous treatment of all of the input/output pairs, than could be extracted by moment methods. It has been shown in Rossen (1972) that, in this case, the minimum partial realization is generally a more accurate representation of the system than can be obtained using moment methods. Thus, if the measured moments contain errors, the minimum realization approach would be expected to generate a more accurate model of the approximate system (that is, the measured moments) by fitting the given measured moments exactly. The model constructed with moment methods is essentially an approximate model of approximate data while the partial realization is a more accurate model of approximate data. The partial realization would then be expected to generate more accurate predictions of future system behavior in response to other forcing functions. This result is reasonable because the minimum realization algorithms were devised specifically for the solution of linear systems while the other methods were developed to treat all varieties of systems.

Another advantage enjoyed by the realization techniques is that the dimension of the state vector need not be selected a priori as in the case of some moment methods. The realization approach determines this dimension mathematically, not intuitively. The dimension of the resulting state-space representation of the physical process will be the minimum dimension which corresponds exactly to the given moments or Markov parameters.

The Bruni adaptation of the Ho-Kalman algorithm is probably the best way to obtain a model for a linear system from the moments, (Rossen, 1972). It should also be noted that the usual result of moment analysis and frequency response analysis of experimental data is the transfer function of the system. To find a set of state-variable equations to fit this transfer function exactly, one can make use of the minimum realization techniques. In this way, the state-variable equations can be found to exactly fit the transfer function. Jeffreson (1970) has recently suggested a method from which the moments of a system can be obtained from the frequency response data. In an attempt to verify the theories of dynamic testing by moment analysis and dynamic testing by frequency response analysis, Jeffreson showed that conventional methods of least squares polynomial curve fitting in the frequency domain allow estimates to be made of the moments of the impulse response of a linear system. From the theory of mathematical statistics, it can be shown that the magnitude and phase angle of the transfer function can be expanded as even and odd power series, respectively, in frequency. The coefficients of the powers of the frequency are easily converted to the central moments of an impulse response. He therefore indicated that the moments required for Ho's algorithm could be obtained from frequency response data by conventional techniques. Frequency response analysis and moment methods, then, can be converted back and forth and the information necessary for the more accurate minimum realization algorithms can be obtained readily from either. Figure 1 shows a schematic for a possible flow of information in model analysis. The most advantageous sequence of steps to be used in the analysis of any linear system may involve, as the final step, a solution using minimum realization algorithms.

EXAMPLE PROBLEM

As an illustration of some of the techniques discussed in this paper, consider the system described by the transfer function

$$\bar{Y}(s) = \begin{bmatrix} \frac{1}{s+.1} & 0 & \frac{2s+.4}{(s+.1)(s+.3)} \\ 0 & \frac{2s+.6}{(s+.4)(s+.2)} & 0 \\ \frac{1}{(s+.5)} & \frac{1}{(s+.2)} & 0 \end{bmatrix}$$

Assume that there exists such a system and that the transfer function $\bar{Y}(s)$ has been determined. Then, a state-space representation of the system can be constructed by any of the procedures detailed in this paper. Using the algorithm developed by Kalman, it can be seen that

$$s_1 = s^2 + .6s + .05$$

$$s_2 = s^2 + .6s + .08$$

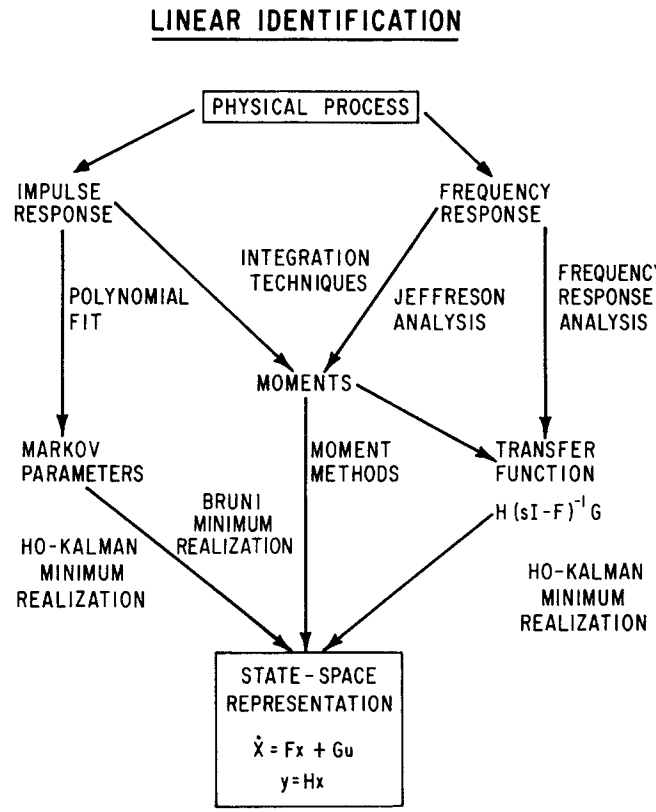

Fig. 1. Schematic of linear system model analysis.

$$s_3 = s^2 + .4s + .03$$

$$n_{11}^* = s + .5$$

$$n_{21}^* = 0$$

$$\rho\mathbf{H}_{23} = \begin{bmatrix} 1 & 0 & 2 & -.1 & 0 & -.4 & .01 & 0 & .1 \\ 0 & 2 & 0 & 0 & -.6 & 0 & 0 & .2 & 0 \\ 1 & 1 & 0 & -.5 & -.2 & 0 & .25 & .04 & 0 \\ -.1 & 0 & -.4 & .01 & 0 & .1 & -.001 & 0 & -.028 \\ 0 & -.6 & 0 & 0 & .2 & 0 & 0 & -.072 & 0 \\ -.5 & -.2 & 0 & .25 & .04 & 0 & -.0625 & -.008 & 0 \end{bmatrix} = 5$$

$$n_{31}^* = s + 1$$

$$n_{12}^* = 0$$

$$n_{22}^* = 2s + .6$$

$$n_{32}^* = s + .4$$

$$n_{13}^* = 2s + .4$$

$$n_{23}^* = n_{33}^* = 0$$

The realization is then

$$\mathbf{H}_c = \begin{bmatrix} .5 & 1 & 0 & 0 & .4 & 2 \\ 0 & 0 & .6 & 2 & 0 & 0 \\ .1 & 1 & .4 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{G}_c = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{F}_c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -.05 & -.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -.08 & -.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -.03 & -.4 \end{bmatrix}$$

By forming the matrices of (14) the reader can see that this realization is controllable but not observable. From this controllable realization, a minimum realization can be found as shown in the discussion of the Mayne algorithm

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.5 & 1 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 3 & 0 & .4 \\ 1 & 0 & 2 \\ 0 & .6 & 0 \\ 0 & 2 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0 & -.03 & 0 & 0 & 0 \\ 1 & -.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.08 & 0 \\ 0 & 0 & 1 & -.6 & 0 \\ 0 & 0 & 0 & -.15 & -.5 \end{bmatrix}$$

This solution is both observable and controllable and is thus of minimum dimension.

2. Assume now, that the transfer function is not available and that the data is in the form of Markov parameters

$$\mathbf{Y}_0 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{Y}_1 = \begin{bmatrix} -.1 & 0 & -.4 \\ 0 & -.6 & 0 \\ -.5 & -.2 & 0 \end{bmatrix}$$

$$\mathbf{Y}_2 = \begin{bmatrix} .01 & 0 & .1 \\ 0 & .2 & 0 \\ .25 & .04 & 0 \end{bmatrix} \quad \mathbf{Y}_3 = \begin{bmatrix} -.001 & 0 & -.028 \\ 0 & -.072 & 0 \\ -.0625 & -.008 & 0 \end{bmatrix}$$

The ranks of the Hankel matrices are (see Tether method)

$$\rho\mathbf{H}_{2,2} = \rho\mathbf{H}_{3,2} = 5$$

so that $n_0 = 5$ and the realization can be constructed from $\mathbf{H}_{2,2}$ by Ho's algorithm. The resulting solution is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} -.1 & 0 & -.2 & .04 & 0 \\ 0 & -.3 & 0 & 0 & .01 \\ .2 & -.05 & -.6 & .02 & -.01 \\ 1 & -.25 & -1.5 & -.2 & -.05 \\ 0 & 1 & 0 & 0 & -.3 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The reader can verify that $\overline{\mathbf{Y}}(s) = \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$.

EXTENSION TO COMPLEX SYSTEMS

In the second paper in this series, techniques for the partial realization of nonlinear lumped parameter systems and distributed parameter systems will be developed. The Tether algorithm will be applied to several varieties of chemical engineering systems. Deficiencies in the Tether approach will be examined and procedures which enhance the effectiveness of the realization method will be outlined. The most critical deficiency in the partial realization approach is the tendency of the Tether algorithm to generate an unstable model based on the input/output data from a stable system. A procedure will be developed to convert these unstable representations into stable representations which will still correspond exactly to the given set of Markov parameters. The application of the Ho algorithm to systems whose behavior is characterized by multiple steady states, unstable steady states, stiff Jacobian matrices, and limit cycles will be discussed. Application of the algorithm to systems described by parabolic, hyperbolic, and elliptic partial differential equations will also be illustrated and examined.

ACKNOWLEDGMENT

The authors wish to acknowledge the support of this work from National Science Foundation Grant NSF-GP-2858. Furthermore, this work made use of Princeton University computer facilities supported in part by National Science Foundation Grant NSF-GJ-34 and GU-3157.

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A Study of Inversion Curves

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A comparative study has been made on the predictive capabilities of a number of the more popular equations of state in use today. The equations were compared in their ability to predict the locus of points for which the Joule-Thomson coefficient is zero, the inversion curve.

The prediction of an inversion curve is an extremely severe test of an equation of state. To date inversion curves have been calculated only for the Van der Waals, Dieterici, Lennard-Jones and Devonshire, and DeBoer-Michels equations of state. This study covers the Van der Waals and Dieterici equations as well as the Virial, Berthelot, Redlich-Kwong, Beattie-Bridgeman, Benedict-Webb-Rubin, and Martin-Hou equations of state.

The results of the investigation show, among other things, that the Redlich-Kwong equation is quite unusual in that it predicts the inversion locus with more accuracy than any of the much more complex equations of state. Its predictive capabilities extend into the liquid region.

Since the classical work of Van der Waals in 1873, numerous equations of state have appeared in the literature. Dodge (*4*) has indicated that the number may well exceed 100. Since this estimate was made more than 26 years ago, the total number now is considerably higher. Of these equations, only a handful have come to be widely known and used.

To date there have been countless numbers of articles concerning the prediction of properties of gases and liquids via the use of an equation of state. Of the many articles that compare the predictive properties of the more common equations, the most notable of recent times is the work of Shah and Thodos (*14*). For the most part, the comparisons of these equations have been made using PVT data as the basis for comparison. The use of this type of data for comparison does not, however, subject the equation of state to as severe a test as could be obtained by the prediction of derived properties, such as heat capacities and Joule-Thomson coefficients.

As has been pointed out by Gunn, Chueh, and Prausnitz (*5*), the prediction of the points at which the Joule-Thomson coefficient is zero, the inversion curve, is an extremely severe test of an equation of state. This test is

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